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MATHEMATICAL MORPHOLOGY AND SPATIAL REASONING: FUZZY AND BIPOLAR SETTING*

ISABELLE BLOCH[1]

Abstract. This paper summarizes the mathematical aspects of fuzzy and bipolar mathematical morphology and shows how morphological operators can be applied to define spatial relations, and then used for spatial reasoning. Fuzzy sets are useful both at representation level and at reasoning level, and contribute to establish links between various areas of artificial intelligence. This is illustrated on the example of spatial reasoning for image understanding. The work on spatial relations was initiated during a sabbatical stay at Berkeley, and this paper was written in memory of Lotfi Zadeh and the enlightening discussions with him and his team.

Keywords: mathematical morphology, lattices, fuzzy sets, bipolar fuzzy sets, fuzzy logic, knowledge representation, spatial relations, spatial reasoning, image understanding.

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1. Introduction

My work on fuzzy mathematical morphology started in the early 1990s, and reading the papers by Lotfi Zadeh was a great source of inspiration, in particular to really see fuzzy sets as sets in the spatial domain. Then, with the aim of exploiting expert knowledge to help guiding segmentation and recognition in images, in particular in the field of medical imaging, I investigated the field of knowledge-based image understanding, and it became soon obvious that spatial relations had to play an important role. However, expert knowledge is often expressed in a linguistic way, and for several relations, no satisfactory mathematical definitions existed. This led me to work on proposing fuzzy models of such relations, mostly based on mathematical morphology. I benefitted from a sabbatical stay at Berkeley (October-November 1995 and January 1997) to work in this direction. Enlightening discussions during seminars chaired by Lotfi Zadeh on various subjects related to the theory of fuzzy sets and the applications thereof, and other nice discussions with several persons of his team, certainly encouraged me to continue to develop this research line. Although no one was working in the field of image understanding and spatial reasoning, I could get inspiration from these seminars and discussions.

This paper summarizes the mathematical aspects of fuzzy and bipolar mathematical morphology and shows how morphological operators can be applied to define spatial relations, and then used for spatial reasoning. Fuzzy sets are useful both at representation level and at reasoning level, and contribute to establish links between various areas of artificial intelligence. This is illustrated on the example of spatial reasoning for image understanding.

The algebraic framework is described in Section 2. It is common to several extensions of mathematical morphology, in particular in a fuzzy setting. Section 3 explains how spatial relations can be defined from mathematical morphology operations, in particular dilations with

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* This work was partly done while the author was with LTCI, Télécom Paris, Institut Polytechnique de Paris France

[1]Sorbonne Université, CNRS, Paris, France
e-mail: isabelle.bloch@sorbonne-universite.fr

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structuring elements representing the semantics of the relations. Section 4 deals with spatial reasoning and image understanding.

2. Fuzzy and bipolar mathematical morphology in complete lattices

Mathematical morphology relies on concepts and tools from various branches of mathematics: algebra (lattice theory), topology, discrete geometry, integral geometry, geometrical probability, partial differential equations, etc. [70, 88]. In this paper only the deterministic part of mathematical morphology and its algebraic setting will be considered. Furthermore, we only provide the definitions and properties of the main basic operators, although many others can be derived, since these operators are the ones that will be used to model structural information and reason on it in Sections 3 and 4.

2.1. **General algebraic setting.** The basic structure in this framework is a complete lattice[1]. Let $(L, \leq)$ and $(L', \leq')$ be two complete lattices. We denote the supremum in L by $\bigvee$ (respectively $\bigvee'$ in L'), the infimum in L by $\bigwedge$ (respectively $\bigwedge'$), the smallest element by 0_L (respectively $0_{L'}$) and the greatest element by 1_L (respectively $1_{L'}$). We have $0_L = \bigwedge L = \bigvee \emptyset$ and $1_L = \bigvee L = \bigwedge \emptyset$, and similar equations in L'. The framework of complete lattices is fundamental in mathematical morphology, as explained in [59, 82, 83].

All the following definitions and results are detailed in textbooks on mathematical morphology, such as [58, 73, 89]. An algebraic dilation is defined as an operator δ from $(L, \leq)$ into $(L', \leq')$ that commutes with the supremum, and an algebraic erosion as an operator ε from $(L', \leq')$ into $(L, \leq)$ that commutes with the infimum, i.e. for every family $(x_i)_{i\in\Xi}$ of elements of L (finite or not), where Ξ is any index set, and for every family $(x'_i)_{i\in\Xi}$ of elements of L', we have:

$$\delta(\bigvee_{i\in\Xi} x_i) = \bigvee_{i\in\Xi}{}' \delta(x_i), \tag{1}$$

$$\varepsilon(\bigwedge_{i\in\Xi}{}' x'_i) = \bigwedge_{i\in\Xi} \varepsilon(x'_i). \tag{2}$$

These are the two main operators, from which a lot of others can be built.

Algebraic dilations and erosions satisfy the following properties:

- $\delta(0_L) = 0_{L'}$ and $\varepsilon(1_{L'}) = 1_L$,
- δ and ε are increasing with respect to the partial ordering on L or L',

Another important concept is the one of adjunction. A pair of operators (ε, δ), $\varepsilon : (L', \leq') \to (L, \leq)$, $\delta : (L, \leq) \to (L', \leq')$, defines an adjunction if:

$$\forall x \in L, \forall y \in L',\ \delta(x) \leq' y \Leftrightarrow x \leq \varepsilon(y). \tag{3}$$

If a pair of operators (ε, δ) defines an adjunction, the following important properties hold:

- $\delta(0_L) = 0_{L'}$ and $\varepsilon(1_{L'}) = 1_L$,
- δ is a dilation and ε is an erosion (in the algebraic sense expressed by Equations 1 and 2), and therefore they are increasing;
- $\delta\varepsilon \leq Id'$, where Id' denotes the identity mapping on L' (i.e. $\delta\varepsilon$ is anti-extensive);
- $Id \leq \varepsilon\delta$, where Id denotes the identity mapping on L (i.e. $\varepsilon\delta$ is extensive);
- $\delta\varepsilon\delta\varepsilon = \delta\varepsilon$ and $\varepsilon\delta\varepsilon\delta = \varepsilon\delta$, i.e. the composition of a dilation and an erosion are idempotent operators ($\delta\varepsilon$ is called a morphological opening and $\varepsilon\delta$ a morphological closing).

The following representation theorem holds: an increasing operator δ is an algebraic dilation iff there is an operator ε such that (ε, δ) is an adjunction; the operator ε is then an algebraic erosion and $\forall x \in L', \varepsilon(x) = \bigvee\{y \in L,\ \delta(y) \leq' x\}$. Similarly, an increasing operator ε is an algebraic erosion iff there is an operator δ such that (ε, δ) is an adjunction; the operator δ is then an algebraic dilation and $\forall x \in L, \delta(x) = \bigwedge'\{y \in L',\ \varepsilon(y) \geq x\}$.

[1] Although mathematical morphology has also been extended to complete semi-lattices and general posets [61], based on the notion of adjunction, in this paper we only consider the case of complete lattices.

Finally, let δ and ε be two increasing operators such that $\delta\varepsilon$ is anti-extensive and $\varepsilon\delta$ is extensive. Then (ε, δ) is an adjunction.

Further properties and derived operators can be found in seminal works such as [58, 88, 89], or in more recent ones [25, 73].

2.2. **Structuring element and morphological dilations and erosions.** Among the numerous examples of complete lattices, one will be particularly interesting for the extensions to fuzzy sets, bipolar fuzzy sets, logics: $(\mathcal{P}(E), \subseteq)$, the set of subsets of a set E, endowed with the set theoretical inclusion. It is a Boolean lattice (i.e. complemented and distributive). The smallest and greatest elements are $0_L = \emptyset$ and $1_L = E$, respectively. Considering dilations and erosions from this lattice in itself, we have $\forall X \in \mathcal{P}(E), \delta(X) = \cup_{x \in X} \delta(\{x\})$.

If E is a vectorial or metric space (e.g. $\mathbb{R}^n$), and if δ and ε are additionally supposed to be invariant under translation, then it can be proved that there exists a subset B, called *structuring element*, such that

$$\delta(X) = \{x \in E \mid \check{B}_x \cap X \neq \emptyset\} \tag{4}$$

and

$$\varepsilon(X) = \{x \in E \mid B_x \subseteq X\}, \tag{5}$$

where B_x denotes the translation of B at point x (i.e. $x + B$), and $\check{B}$ is the symmetrical of B with respect to the origin of space. The operators are then called morphological dilations and erosions. Details on these definitions and their properties can be found e.g. in [25, 58, 73, 88].

The structuring element B defines a neighborhood that is considered at each point. This is typically the case in image processing and computer vision, where the underlying lattice is built on sets or functions of the spatial domain. It is a subset of E with fixed shape and size, directly influencing the extent of the morphological operations. It is generally assumed to be compact, so as to guarantee good properties. In the discrete case, it is often defined as a connected subset, according to a discrete connectivity defined on E.

The general principle underlying morphological operators consists in translating the structuring element at every position in space and checking if this translated structuring element satisfies some relation with the original set (intersection for dilation, Equation 4, inclusion for erosion, Equation 5) [88].

An example on a binary image is displayed in Fig.1.

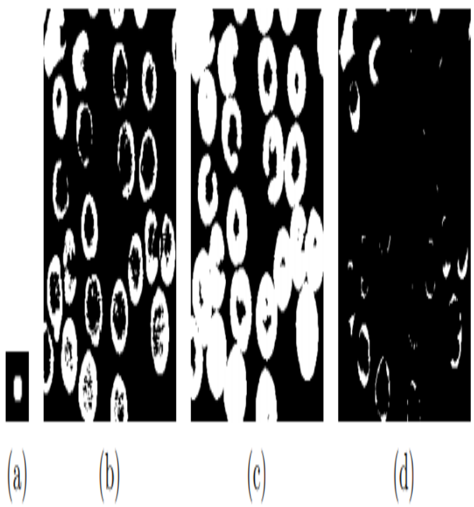

Figure 1. (a) Structuring element B (ball of the Euclidean distance). (b) Subset X in the Euclidean plane (in white). (c) Its dilation $\delta_B(X)$. (d) Its erosion $\varepsilon_B(X)$.

More generally, the structuring element can be seen as a binary relation between elements of E [25], i.e. $y \in B_x$ iff $R(x, y)$ where R denotes a binary relation on $E \times E$. Dilation and erosion are then expressed as follows:

$$\delta(X) = \{x \in E \mid \exists y \in X, R(y, x)\},$$

$$\varepsilon(X) = \{x \in E \mid \forall y \in E, R(x, y) \Rightarrow y \in X\}.$$

These formulas apply for any binary relation R. If R is reflexive (i.e. $R(x, x)$ for all x), then δ is extensive ($X \subseteq \delta(X)$) and ε is anti-extensive ($\varepsilon(X) \subseteq X$). These properties hold in the case illustrated in Fig.1. The objects in the original image are then expanded by dilation, to an extent that depends on the shape and the size of the structuring element, and reduced by erosion. Similar interpretations hold for any relation R, and these properties will also be important in the remainder of this paper.

This view of a structuring element as a binary relation allows applying morphological operators to many different mathematical objects endowed with a lattice structure (e.g. graphs and hypergraphs, logics, fuzzy sets...).

2.3. **Mathematical morphology on fuzzy sets.** Extending mathematical morphology to fuzzy sets was addressed by several authors in the 1990's. Some definitions just consider gray levels as membership degrees, or use binary structuring elements. Here we restrict ourselves to really fuzzy approaches, where fuzzy sets have to be transformed according to fuzzy structuring elements [29]. The main approaches are summarized and compared in [16]. Lattice theory has become a popular mathematical framework in different domains of information processing, such as mathematical morphology, fuzzy sets, formal concept analysis, among others. Extending mathematical morphology to fuzzy sets in this framework allows one to deal with imprecision and vagueness in knowledge representation and information processing, while benefiting from the strong properties induced by the algebraic framework.

Let $\mathcal{F}$ be the set of fuzzy sets defined over a domain $\mathcal{S}$ and with membership values in $[0, 1]$ (this interval is arbitrary, though very usual, and any lattice could be used as well). We identify a fuzzy set with its membership function. Let $\leq$ denote the classical ordering on fuzzy sets, defined as:

$$\forall (\mu, \nu) \in \mathcal{F}^2, \mu \leq \nu \Leftrightarrow \forall x \in \mathcal{S}, \mu(x) \leq \nu(x). \tag{6}$$

Note that the same symbol is used here for the ordering on $[0, 1]$ and on $\mathcal{F}$ (without ambiguity).

It is easy to see that $(\mathcal{F}, \leq)$ is a complete lattice. The least element is the "empty" fuzzy set $\mu_\emptyset$ ($\forall x \in \mathcal{S}, \mu_\emptyset(x) = 0$). The greatest element is the "full" fuzzy set $\mu_\mathcal{S}$ ($\forall x \in \mathcal{S}, \mu_\mathcal{S}(x) = 1$). The infimum and supremum of a family $(\mu_i)_{i \in \Xi}$ where Ξ is an index set, are:

$$(\inf_{i \in \Xi} \mu_i)(x) = \inf_{i \in \Xi}(\mu_i(x)),$$

$$(\sup_{i \in \Xi} \mu_i)(x) = \sup_{i \in \Xi}(\mu_i(x)).$$

Note that again the same notations are used over $[0, 1]$ and over $\mathcal{F}$. The inf and sup operators become min and max in particular on finite families.

Moreover, $(\mathcal{F}, \leq, \inf, \sup, t, I)$ where t and I are adjoint conjunction and implication (i.e. I is the residuated implication derived from t) is a residuated lattice.

Once the lattice is defined, algebraic dilation and erosion are defined as in the general algebraic setting, as well as the derived operators (e.g. opening and closing). They have then the same properties.

Moving now to morphological dilations and erosions, a fuzzy structuring element ν is defined as a fuzzy set in $\mathcal{S}$, or, equivalently, as a fuzzy binary relation in $\mathcal{S} \times \mathcal{S}$. Translating Equations 4 and 5 amongs to replace intersection by a degree of intersection, and inclusion by a degree of implication, leading to the following definitions of fuzzy dilation and fuzzy erosion of a fuzzy set μ by a fuzzy structuring element ν:

$$\forall x \in \mathcal{S},\ \delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} t(\nu(x,y), \mu(y)), \tag{7}$$

where t is a t-norm, and

$$\forall x \in \mathcal{S},\ \varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} I(\nu(y,x)), \mu(y)), \tag{8}$$

where I is the residuated implication of t.

A very important property is that the condition of t and I being adjoint is necessary and sufficient for (ε, δ) to be an adjunction. Other choices of t and I can be made, at the price of loosing some usual properties of mathematical morphology operators.

As in the general setting, opening and closing are morphological filters, i.e. increasing and idempotent operators. They can be further combined to define new filters. A typical example is to use an increasing sequence of structuring elements $\nu_i, i = 1...k$, and apply sequentially opening and closing with each of these structuring elements, thus defining an alternate sequential filter. This classical notion directly extends to fuzzy sets using the previous definitions. By denoting $\gamma_i = \delta_{\nu_i}\varepsilon_{\nu_i}$ and $\varphi_i = \varepsilon_{\nu_i}\delta_{\nu_i}$, the result of the filter applied to a fuzzy set μ is computed as $ASF(\mu) = \gamma_k\varphi_k....\gamma_1\varphi_1(\mu)$ or $ASF(\mu) = \varphi_k\gamma_k....\varphi_1\gamma_1(\mu)$. The filter can be further improved by adding reconstruction steps after each operation, providing more robustness. Fig.2. illustrates the result of an alternate sequential filter with and without reconstruction on a retina image. High gray levels correspond to high membership degrees to the vessels, but the representation is noisy. The noise suppression effect is clearly visible. Only a few traces of vessels are visible on the result of the filter without reconstruction, while their details are recovered by the reconstruction steps. Note also that the contours of the preserved structures are not smoothed.

2.4. **Mathematical morphology on bipolar fuzzy sets.** Bipolarity corresponds to a recent trend in contemporary information processing, both from a knowledge representation point of view, and from a processing and reasoning one. It allows distinguishing between (i) positive information, which represents what is guaranteed to be possible, for instance because it has already been observed or experienced, and (ii) negative information, which represents what is impossible or forbidden, or surely false [46]. This domain has recently motivated work in several directions, for instance for applications in knowledge representation, preference modeling, argumentation, multi-criteria decision analysis, cooperative games, among others [46]. Three types of bipolarity are distinguished in [51]: (i) symmetric univariate, where a unique totally ordered scale covers the range from negative (not satisfactory) to positive (satisfactory) information (e.g. modeled by probabilities); (ii) symmetric bivariate, where two separate scales are linked together and concern related information (e.g. modeled by belief functions); (iii) asymmetric or heterogeneous, where two types of information are not necessarily linked together and may come from different sources. This last type is particularly interesting in image interpretation and spatial reasoning, and the bipolar nature of information has also to be combined with its imprecision or vagueness, leading to the notion of bipolar fuzzy sets.

Let us assume that bipolar information is represented by a pair (μ, ν), where μ represents the positive information and ν the negative information, under a consistency constraint [51], which guarantees that the positive information is compatible with the constraints or rules expressed by the negative information. From a formal point of view, bipolar information can be represented in different settings. Here we consider the representation where μ and ν are membership functions to fuzzy sets, defined over a space $\mathcal{S}$ (for instance the spatial domain, a set of potential options in preference modeling...), and the consistency constraint is expressed as $\forall x \in \mathcal{S}, \mu(x) + \nu(x) \leq 1$. The set of all bipolar fuzzy sets over $\mathcal{S}$ is denoted by $\mathcal{B}$. The pair (μ, ν) is then called a bipolar fuzzy set. For each point x, $\mu(x)$ defines the membership degree of x (positive information) and $\nu(x)$ its non-membership degree (negative information). This formalism allows representing both bipolarity and fuzziness.

As noticed e.g. in [48] and the subsequent discussion, bipolar fuzzy sets are formally equivalent (but with important differences in their semantics) to interval-valued fuzzy sets originally proposed in [100], where the membership of x is expressed (using the same notations) as an interval $[\mu(x), 1-\nu(x)]$ of $[0,1]$ (hence implying the consistency constraint), and to intuitionistic fuzzy sets, where this consistency constraint was also proposed, along with the notion of membership and non-membership degrees [4]. These sets can also be interpreted as thick sets [44], where an ill-known set (or fuzzy set) is represented as an interval of sets (or fuzzy sets). All these are also special cases of L-fuzzy sets introduced in [57]. Despite these formal equivalences, since the semantics are very different, as discussed e.g. in [48, 49], we keep here the terminology of bipolarity.

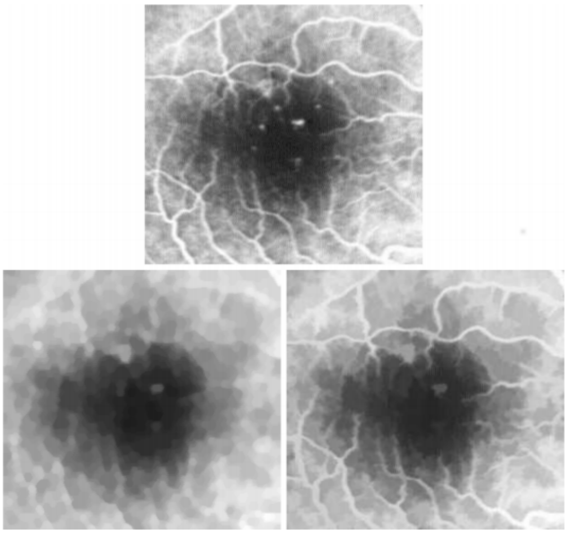

Figure 2. Image of retina vessels on top (high gray levels represent high membership values to the vessels), and below its filtering using alternate sequential filter without reconstruction (left) and with reconstruction (right).

Let us denote by $\mathcal{L}$ the set of ordered pairs of numbers (a,b) in $[0,1]$ such that $a+b \leq 1$ (hence $(\mu,\nu) \in \mathcal{B} \Leftrightarrow \forall x \in \mathcal{S}, (\mu(x),\nu(x)) \in \mathcal{L}$). In all what follows, for each $(\mu,\nu) \in \mathcal{B}$, we will note $(\mu,\nu)(x) = (\mu(x),\nu(x))$ $(\in \mathcal{L})$, $\forall x \in \mathcal{S}$. Note that fuzzy sets can be considered as particular cases of bipolar fuzzy sets, either when $\forall x \in \mathcal{S}, \nu(x) = 1-\mu(x)$, or when only one information is available, i.e. $(\mu(x), 0)$ or $(0, 1-\mu(x))$. Furthermore, if μ (and ν) only takes values 0 and 1, then bipolar fuzzy sets reduce to classical sets.

Note that the interval $[0,1]$ is an arbitrary choice, and a more general setting could rely on L-fuzzy sets [57], by only assuming that $\mathcal{L}$ is a poset or a complete lattice. This line was followed in [91] for instance. Here we have chosen to keep $\mathcal{L}$ as presented above for simplifying the presentation and highlighting the bipolar nature of the information with its two components, but all what follows in this section and in the next one actually applies to more general forms of $\mathcal{L}$.

Let $\preceq$ be a partial ordering on $\mathcal{L}$ such that $(\mathcal{L},\preceq)$ is a complete lattice. We denote by $\bigvee$ and $\bigwedge$ the supremum and infimum, respectively. The smallest element is denoted by $0_{\mathcal{L}}$ and the largest element by $1_{\mathcal{L}}$. The partial ordering on $\mathcal{L}$ induces a partial ordering on $\mathcal{B}$, also denoted by $\preceq$ for the sake of simplicity:

$$(\mu_1,\nu_1) \preceq (\mu_2,\nu_2) \text{ iff } \forall x \in \mathcal{S}, (\mu_1,\nu_1)(x) \preceq (\mu_2,\nu_2)(x). \tag{9}$$

Then $(\mathcal{B},\preceq)$ is a complete lattice, for which the supremum and infimum are also denoted by $\bigvee$ and $\bigwedge$. For any family $\{(\mu_i,\nu_i), i \in \Xi\}$ of elements of $\mathcal{B}$, where Ξ is an index set, we note

$(\bigwedge_{i\in\Xi}(\mu_i,\nu_i))(x) = \bigwedge_{i\in\Xi}(\mu_i(x),\nu_i(x))$ and $(\bigvee_{i\in\Xi}(\mu_i,\nu_i))(x) = \bigvee_{i\in\Xi}(\mu_i(x),\nu_i(x))$, for all x in $\mathcal{S}$. The smallest element of $(\mathcal{B},\preceq)$ is the bipolar fuzzy set $0_{\mathcal{B}} = (\mu_0,\nu_0)$ taking value $0_{\mathcal{L}}$ at each point, and the largest element is the bipolar fuzzy set $1_{\mathcal{B}} = (\mu_{\mathbb{I}},\nu_{\mathbb{I}})$ always equal to $1_{\mathcal{L}}$.

In this framework, connectives are defined as direct extensions of their fuzzy counterpart [43]. In particular, adjoint bipolar conjunction and implication will be useful to derive mathematical morphology operators on bipolar fuzzy sets, guaranteeing the preservation of the usual properties.

While an operation between two bipolar fuzzy sets (their intersection for example) can be computed using bipolar connectives (a conjunction for example, applied to every element of the support of the bipolar fuzzy sets), providing a new bipolar fuzzy set, here we want to assess to which degree a relation is satisfied by two bipolar fuzzy sets (e.g. two which degree they intersect), which is a different question. In the classical setting, this degree would be 0 or 1, and in the fuzzy setting, it would be a number in $[0,1]$. Here, inclusion and intersection are defined as bipolar numbers (i.e. in $\mathcal{L}$) (see e.g. [19] and the references therein), so as to keep track of the imprecise and bipolar nature of information. A bipolar degree of inclusion of (μ_1,ν_1) in (μ_2,ν_2) is defined from a bipolar implication I as:

$$Inc((\mu_1,\nu_1),(\mu_2,\nu_2)) = \bigwedge_{x\in\mathcal{S}} I((\mu_1,\nu_1)(x),(\mu_2,\nu_2)(x)). \tag{10}$$

A bipolar degree of intersection of (μ_1,ν_1) and (μ_2,ν_2) is defined from a bipolar conjunction C as:

$$Int((\mu_1,\nu_1),(\mu_2,\nu_2)) = \bigvee_{x\in\mathcal{S}} C((\mu_1,\nu_1)(x),(\mu_2,\nu_2)(x)). \tag{11}$$

These definitions are direct extensions of the degrees of inclusion and intersection of fuzzy sets, defined as the infimum of a fuzzy implication and the supremum of a fuzzy conjunction, respectively.

Mathematical morphology on bipolar fuzzy sets was proposed for the first time in [14], by considering the complete lattice defined from the Pareto ordering. Then it was further developed, with additional properties, geometric aspects and applications to spatial reasoning, in [15, 17]. The lexicographic ordering was considered too in [18]. In [19], any partial ordering was considered, and derived operators were also proposed. Similar work has been developed independently, in the setting of intuitionistic fuzzy sets and interval-valued fuzzy sets, also based on Pareto ordering (e.g. [72]). This group proposed an extension to L-fuzzy sets [91], besides its important contribution to connectives (e.g. [43]). The general algebraic framework of mathematical morphology, where a dilation is an operator that commutes with the supremum and an erosion an operator that commutes with the infimum applies here as well, considering the lattice $(\mathcal{B},\preceq)$. Their particular form involving structuring elements is based on the definition of a structuring element as a binary bipolar relation (μ_B,ν_B) between elements of $\mathcal{S}$. Its value $(\mu_B,\nu_B)(x,y)$, for $x\in\mathcal{S}, y\in\mathcal{S}$, represents the bipolar degree to which this relation is satisfied between x and y. If $\mathcal{S}$ is endowed with a translation (for instance if it is a subset of $\mathbb{R}^n$ or $\mathbb{Z}^n$, representing a spatial domain), then a typical example would be $(\mu_B,\nu_B)(x,y) = (\mu_B,\nu_B)(y-x)$, and the value of a structuring element at $y-x$ represents the value at point y of the translation of the structuring element at point x. The erosion of any (μ,ν) in $\mathcal{B}$ by (μ_B,ν_B) is defined from a bipolar implication I as:

$$\forall x\in\mathcal{S},\ \varepsilon_{(\mu_B,\nu_B)}((\mu,\nu))(x) = \bigwedge_{y\in\mathcal{S}} I((\mu_B,\nu_B)(x,y),(\mu,\nu)(y)). \tag{12}$$

The dilation of any (μ,ν) in $\mathcal{B}$ by (μ_B,ν_B) is defined from a bipolar conjunction C as:

$$\forall x\in\mathcal{S},\ \delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) = \bigvee_{y\in\mathcal{S}} C((\mu_B,\nu_B)(y,x),(\mu,\nu)(y)). \tag{13}$$

These definitions are proved to provide bipolar fuzzy sets, and express erosion (respectively dilation), as a degree of inclusion according to Equation 10 (respectively intersection, Equation 11) of the translation (if defined on $\mathcal{S}$) of the structuring element and the bipolar fuzzy set to be transformed.

The properties of these definitions are detailed in [19], and are direct extensions from the classical properties on sets or fuzzy sets. They are not recalled here.

An illustration of a simple bipolar fuzzy set and its dilation with a bipolar fuzzy structuring element is displayed in Fig.3.

2.5. **Morpho-logics.** For reasoning purpose, mathematical morphology can also be applied to logical formulas. This idea has been first introduced in [26, 27] for propositional logic. Let PS be a finite set of propositional symbols, with $|PS| = N$. The set of formulas (generated by PS and the usual connectives) is denoted by Φ. Well-formed formulas are denoted by Greek letters φ, ψ... The set of all interpretations for Φ is denoted by $\Omega = \mathbf{2}^{|PS|}$, interpretations are denoted by ω, ω'..., and $\|\varphi\| = \{\omega \in \Omega \mid \omega \models \varphi\}$ is the set of all models of φ (i.e. all interpretations for which φ is true).

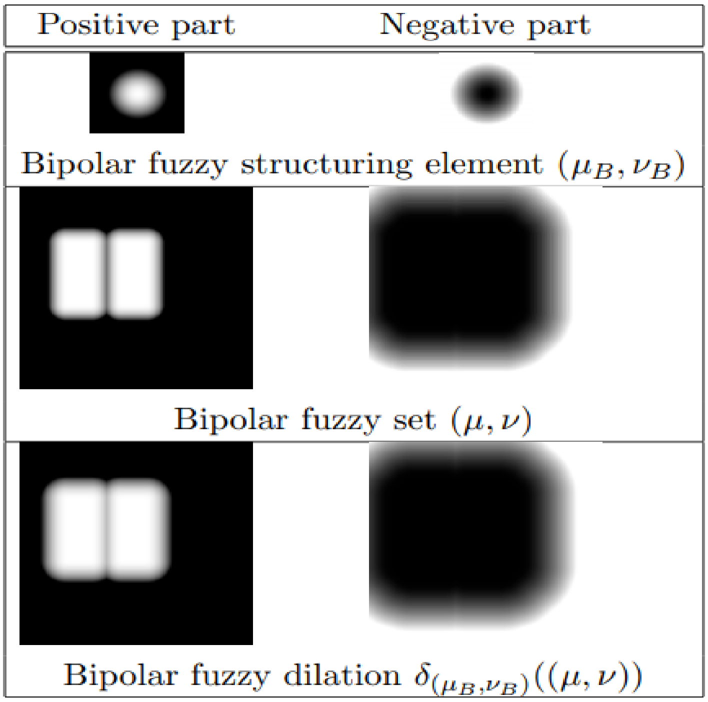

Figure 3. Illustration of a simple bipolar fuzzy dilation. High grey levels represent high membership values. Since the origin completely belongs to the structuring element (with bipolar degree equal to $1_{\mathcal{L}}$), the dilation is extensive (i.e. $(\mu, \nu) \preceq \delta_{(\mu_B,\nu_B)}((\mu, \nu))$), which is illustrated by a larger positive part and a reduced negative part.

The underlying idea for constructing morphological operations on logical formulas is to consider formulas and interpretations from a set theoretical perspective. Since Φ is isomorphic to $\mathbf{2}^{\Omega}$ up to the syntactic equivalence, i.e., knowing a formula defines completely the set of its models (and conversely, any set of models corresponds to a subset of Φ built of syntactic equivalent formulas), we can identify φ with the set of its models $[\varphi]$, and then apply set-theoretic morphological operations.

We recall that $[\varphi \vee \psi] = [\varphi] \cup [psi]$, $[\varphi \wedge \psi] = [\varphi] \cap [\psi]$, $[\varphi] \subseteq [\psi]$ iff $\varphi \models \psi$, and φ is consistent iff $[\varphi] \neq \emptyset$. Considering the inclusion relation on $\mathbf{2}^{\Omega}$, $(\mathbf{2}^{\Omega}, \subseteq)$ is a Boolean complete lattice. Similarly a lattice (which is isomorphic to $\mathbf{2}^{\Omega}$) is defined on $\Phi_{\equiv}$, where $\Phi_{\equiv}$ denotes the quotient space of Φ by the equivalence relation between formulas (with the equivalence defined as $\varphi \equiv \psi$ iff $[\varphi] = [\psi]$). In the following, this is implicitly assumed, and we simply use the notation Φ. Any subset $\{\varphi_i\}$ of Φ has a supremum $\bigvee_i \varphi_i$, and an infimum $\bigwedge_i \varphi_i$ (corresponding respectively to union and intersection in $\mathbf{2}^\Omega$). The greatest element is $\top$ and the smallest one is $\perp$ (corresponding respectively to $\mathbf{2}^\Omega$ and $\emptyset$).

Based on this lattice structure, it is straightforward to define a dilation as an operation that commutes with the supremum and an erosion as an operation that commutes with the infimum. They naturally inherit all properties of the general algebraic framework.

Using the previous equivalences, morphological dilation and erosion of a formula with a structuring element are then defined. The underlying lattice is $(\Phi_\equiv, \models)$, or equivalently $(\mathbf{2}^\Omega, \subseteq)$. Since these two lattices are isomorphic, we will use the same notations for morphological operations on each of them. A morphological dilation of a formula φ with a structuring element B ($B \in \mathbf{2}^\Omega$) is defined through its models as:

$$[\delta_B(\varphi)] = \delta_B([\varphi]) = \{\omega \in \Omega \mid \check{B}_\omega \wedge \varphi \; consistent\}. \tag{14}$$

Similarly, a morphological erosion is defined as:

$$[\varepsilon_B(\varphi)] = \varepsilon_B([\varphi]) = \{\omega \in \Omega \mid B_\omega \models \varphi\}. \tag{15}$$

In these equations, the structuring element B represents a relationship between worlds, i.e. $\omega' \in B_\omega$ iff ω' satisfies some relationship with ω. The condition in Equation 14 expresses that the set of worlds in relation to ω should be consistent with φ. The condition in Equation 15 is stronger and expresses that all worlds in relation to ω should be models of φ. Again properties are the same as in the classical setting.

These definitions can be extended to other types of logics. In particular, a direct extension to fuzzy logic is obtained by considering that the set of models of a formula is a fuzzy subset of Ω. A bipolar component can be added as well, by considering the set of models of a formula as a bipolar fuzzy subset of Ω. The definitions of δ and ε in a semantic way (via the sets of models) make these extensions rather straightforward, using the definitions in Section 2.3 and 2.4.

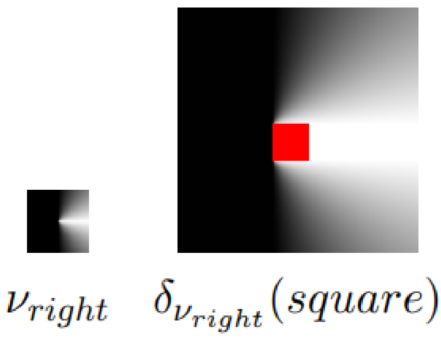

Figure 4. Fuzzy set representing the semantics of to the right of", and dilation providing the region of space to the right of the red square. Gray-values represented membership degrees ($black = 0, white = 1$).

3. Mathematical modeling of spatial relations

Spatial reasoning, as will be seen in Section 4, strongly relies on structural information, especially on the spatial arrangement of the objects in a scene. Modeling spatial relations is then a key feature in this field. When imprecision on knowledge and on data has to be taken into account, a semi-qualitative framework is best appropriate, such as fuzzy sets theory. In particular, it allows designing mathematical models when objects are imprecisely defined, when spatial relations themselves are intrinsically vague or imprecise (such as "to the left of"), or when they are described as concepts, in a linguistic or symbolic way (e.g. "close to"). One of the problems to be addressed when reasoning on both qualitative or symbolic knowledge and on numerical data is the so-called semantic gap, between abstract concepts and concrete information extracted from data (e.g. images). The notion of linguistic variable [100] is then useful to establish links between a concept and its representation in a specific concrete domain. The usefulness of fuzzy representations of spatial relations was already advocated in the 1970's [54]. Different classes of spatial relations were identified e.g. in [62], in particular the two important classes of topological and metric relations. Here again fuzzy mathematical morphology can play an important role for modeling spatial relations [12].

This work has started during my sabbatical stay at Berkeley, and led to a join work on adjacency [30]. Then I continued to work on the modeling of spatial relations, in a fuzzy set setting, in a logical framework, and more recently in the framework of bipolar fuzzy sets.

Relations that are well defined when objects are crisp can be extended to the fuzzy case by translating their mathematical expressions, replacing each element by its fuzzy counterpart (e.g. a set by a fuzzy set, an intersection by a fuzzy conjunction...). This leads to direct extensions of these relations, that preserve their properties. If the relations are not well defined, then a direct fuzzy formulation should be designed. The main idea is to model the semantics of a spatial relation R expressed in a linguistic way as a fuzzy structuring element ν_R (i.e. a function from the spatial domain into $[0, 1]$). Then dilating a reference object, possibly fuzzy, defined by its membership function μ, provides a fuzzy region of space, where the value $\delta_{\nu_R}(\mu)(x)$ at each point x represents the degree to which relation R to μ is satisfied at this point. The degree to which another object satisfies relation R to μ can then be computed from the aggregation, using some fuzzy fusion operator, of the values $\delta_{\nu_R}(\mu)(x)$ for all points x of this second object. Several relations can be modeled according to this principle, such as topological relations, directions, distances, as well as more complex relations (between, along, parallel, aligned...). The relation "to the right of" is illustrated in Fig.4.

A useful feature of fuzzy representations is that a relation and its degree of satisfaction can be represented as a number, a fuzzy number, an interval, a distribution, a spatial fuzzy set, etc., in a same unifying framework. Moreover, of bipolarity has additionally to be taken into account, a similar way of reasoning can be applied to define bipolar spatial relations. This is not detailed here, and one may refer to [17, 21].

Another feature of fuzzy set theory can be found in its algebraic structure, that merges nicely with the algebraic component of mathematical morphology. This can be exploited by expressing spatial relations in an algebraic way.

Let us detail an example for distance relations [11]. Distance between objects is an important information for the assessment of spatial arrangement between objects in a scene. Therefore they are widely used in structural pattern recognition and spatial reasoning. Distances between objects A and B can be expressed in different forms, using linguistic expressions such as *the distance between A and B is equal to n*, *the distance between A and B is less (respectively greater) than n*, *the distance between A and B is between n_1 and n_2*. These expressions are then translated in spatial regions or volumes of interest within the spatial domain $\mathcal{S}$, taking into account imprecision and uncertainty, since these statements are generally approximate (n, n_1, n_2 can be numbers, but also intervals, fuzzy numbers, linguistic values, etc.). Distances between sets (average, Hausdorff, minimum distances) are usually defined by analytical expressions. But they also have equivalents in set theoretical terms by means of mathematical morphology. For instance, the minimum (respectively Hausdorff) distance between two sets is the minimal size of dilation to be applied to one set so that it intersects (respectively includes) the other set. This allows to easily include imprecision, and to deal with distances between fuzzy sets and with fuzzy distances. Let us assume that A is known as one already recognized object, or a known area of $\mathcal{S}$, and let us determine B, subject to satisfy some distance relationship with A. According to the morphological expressions of distances, dilation of A is an adequate tool for this. Let us consider the following different cases, where $\delta^n(A)$ denotes the dilation of A by a structuring element of size n:

- If knowledge expresses that $d(A, B) = n$, then the border of B should intersect the region defined by $\delta^n(A) \setminus \delta^{n-1}(A)$, which is made up of the points exactly at distance n from A, and B should be looked for in $\delta^{n-1}(A)^C$ (the complement of the dilation of size $n-1$).
- If knowledge expresses that $d(A, B) \leq n$, then B should be looked for in A^C, with the constraints that at least one point of B belongs to $\delta^n(A) \setminus A$.
- If knowledge expresses that $d(A, B) \geq n$, then B should be looked for in $\delta^{n-1}(A)^C$.
- If knowledge expresses that $n_1 \leq d(A, B) \leq n_2$, then B should be searched in $\delta^{n_1-1}(A)^C$ with the constraint that at least one point of B belongs to $\delta^{n_2}(A) \setminus \delta^{n_1-1}(A)$.

The constraints on the border lead to the definition of actually two fuzzy sets, one for constraining the object, and one constraining its border, as for adjacency. However, they can be avoided by considering both minimum and maximum (Hausdorff) distances, expressing for instance that B should lay between a distance n_1 and a distance n_2 of A. Therefore, the minimum distance should be greater than n_1 and the maximum distance should be less than n_2. In this case, the volume of interest for B is reduced to $\delta^{n_2}(A) \setminus \delta^{n_1-1}(A)$.

In cases where imprecision has to be taken into account, fuzzy dilations are used, with the corresponding equivalences with fuzzy distances [10, 29]. The extension to approximate distances calls for fuzzy structuring elements, defined through their membership function ν on $\mathcal{S}$. Structuring elements with a spherical symmetry can typically be used, where the membership degree only depends on the distance to the center of the structuring element. Let us consider the generalization to the fuzzy case of the last case (minimum distance of at least n_1 and maximum distance of at most n_2 to a fuzzy set μ). Instead of defining an interval $[n_1, n_2]$, we consider a fuzzy interval, defined as a fuzzy set on $\mathbb{R}^+$ having a core equal to the interval $[n_1, n_2]$. The membership function μ_n is increasing between 0 and n_1 and decreasing after n_2. Note that for a specific application, these parameters can be learned easily from annotated data [6]. Then two structuring elements are derived, as:

$$\nu_1(v) = \begin{cases} 1 - \mu_n(d_E(v, 0)) & \text{if } d_E(v, 0) \leq n_1 \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

$$\nu_2(v) = \begin{cases} 1 & \text{if } d_E(v, 0) \leq n_2 \\ \mu_n(d_E(v, 0)) & \text{otherwise} \end{cases} \tag{17}$$

where d_E is the Euclidean distance in $\mathcal{S}$ and O the origin of space. The spatial fuzzy set expressing the approximate relationship about distance to μ is then defined as:

$$\mu_{dist} = \begin{cases} t[\delta_{\nu_2}(\mu), 1 - \delta_{\nu_1}(\mu)] & \text{if } n_1 \neq 0 \\ \delta_{\nu_2}(\mu) & \text{if } n_1 = 0 \end{cases} \tag{18}$$

where t is a t-norm. The increasingness of fuzzy dilation with respect to both the set to be dilated and the structuring element [29] guarantees that these expressions do not lead to inconsistencies. Indeed, we have $\nu_1 \leq \nu_2$, $\nu_1(0) = \nu_2(0) = 1$, and therefore $\mu \leq \delta_{\nu_1}(\mu) \leq \delta_{\nu_2}(\mu)$. In the case where $n_1 = 0$, we do not have $\nu_1(0) = 1$ any longer, but in this case, only the dilation by ν_2 is considered. This case corresponds actually to a distance to μ "less than about n_2". These properties are indeed expected for representations of distance knowledge.

Fig.5 illustrates this approach. The two structuring elements ν_1 and ν_2 are derived from a fuzzy interval μ_n, are used for dilation of an object, and μ_{dist} is computed to represent the approximate knowledge about the distance to this object. This example also illustrates the usefulness of linguistic variables. Here the linguistic value of the distance (e.g. medium distance) is converted as a fuzzy set on the real line (one concrete domain), providing the semantics of this value, and then as structuring elements in the image space (another concrete domain).

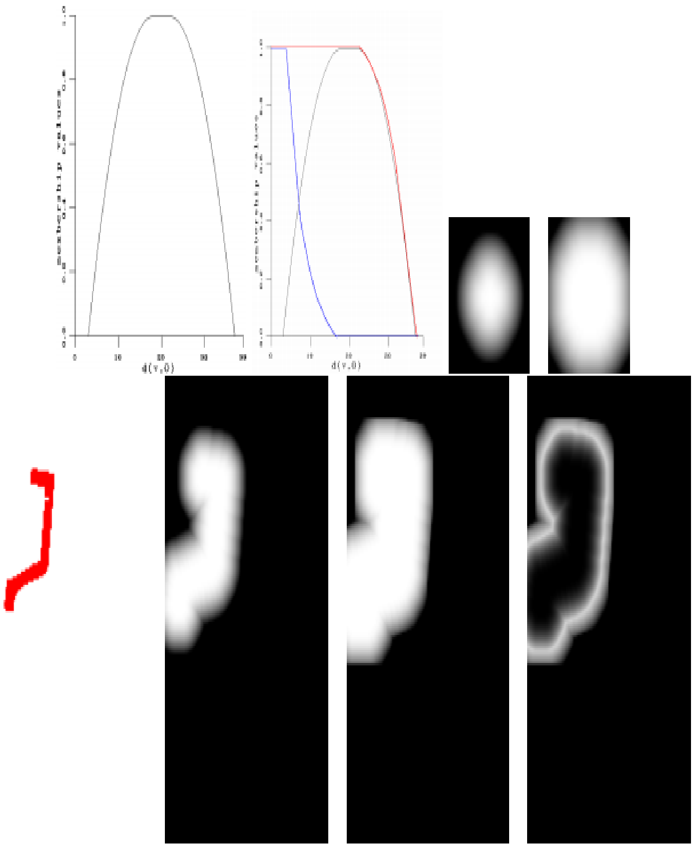

Figure 5. Top: Membership function μ_n, structuring elements ν_1 and ν_2 represented on the real line and then in the spatial domain. Bottom: Reference object, dilations with these two structuring elements, and representation of μ_{dist}, i.e. the fuzzy region of space at the desired distance from the reference object.

From an algorithmic point of view, fuzzy dilations may be quite heavy if the structuring element has a large support. However, in the case of crisp objects and structuring elements with spherical symmetry, fast algorithms can be implemented. The distance to the object A is first computed using chamfer algorithms [34]. It defines a distance map in $\mathcal{S}$, which gives the distance of each voxel v to object A, and that can be as precise as necessary. Then the translation into a fuzzy volume of interest is made according to a simple look-up table derived from μ_n. This algorithm has a linear complexity in the cardinality of $\mathcal{S}$.

This framework can be the basis of spatial reasoning, for example for structural model based image understanding. Indeed, modeling explicitly the structural knowledge we may have on a scene helps recognizing individual structures (disambiguating them in case of similar shape and appearance), as well as their global organization. This can be achieved using various methods, each having two main components: knowledge representation and reasoning. Fuzzy models of objects and relations can enhance qualitative representations (logical formulas, knowledge bases, ontologies), and then be used in logical reasoning, including morpho-logic (for instance, the set of models of a formula becomes a fuzzy set). They can serve as attributes in structural representations such as graphs, hypergraphs, conceptual graphs. Reasoning then rely on matching, sequential graph traversal to guide the exploration of an image, constraint satisfaction problems, etc. (see e.g. [20, 22] and the references therein). From the interpretation results, it is then possible to go back to the initial language of the domain to provide linguistic descriptions of the image content, as in the previous example on brain image interpretation. This is further detailed next.

4. Mathematical morphology and spatial reasoning

In this section, we illustrate how mathematical morphology can be used for spatial reasoning in various settings. Spatial reasoning is defined as the domain of knowledge representation on spatial entities and spatial relationships, and reasoning on them. Spatial entities can be represented as abstract formulas in a logical (symbolic, qualitative) setting, as regions or keypoints in a quantitative setting, or as fuzzy regions in the semi-qualitative setting of fuzzy sets. On the symbolic side, spatial relations can be represented as formulas, connectives, modalities or predicates in a logical setting. On the numerical side, they are best represented using fuzzy models, in order to account for their intrinsic imprecision (e.g. "to the right of", "close to"), as explained in Section 3. Both sides often need to be combined, which further enhances the usefulness of the semi-qualitative framework of fuzzy sets.

4.1. **Logics.** With the help of the operators of dilation and erosion, we can address successfully some important reasoning aspects in AI, such as revision, merging (or fusion), abduction (explanatory reasoning), mediation..., which can find very concrete solutions in this morphological framework [28, 31, 32, 33]. Another interesting feature is that these solutions, while simple and tractable, satisfy the properties usually required in such reasoning problems. For instance, revision can be obtained by a minimal dilation of the old information until it becomes consistent with the new one. Fusion of formulas (expressing, in a spatial reasoning context, spatial regions, preferences about positions, etc.) can be obtained by dilating each formula until a consensus is reached (consistency of the dilated formulas). In abductive reasoning, in order to find the "best" explanation to an observation according to a knowledge base, one may erode this base as much as possible, provided it is still consistent with the observation. In all these examples, the minimality constraint, often required, is automatically satisfied. Moreover, the definitions can be expressed semantically, and operators are applied on the models of the formulas. This approach directly applies to fuzzy logic, where the set of models of a formula is a fuzzy set. Extensions to several other logics, more expressive than propositional logic have been developed as well.

An example illustrating the idea of fusion and explanatory reasoning, by representing models of formulas as sets, is provided in Fig.6.

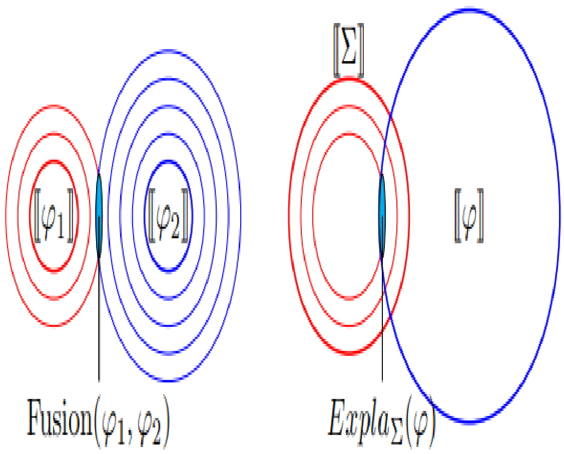

Figure 6. Fusion of two formulas φ_1 and φ_2, based on dilations (left), and explanation of an observation φ according to a knowledge base Σ based on erosions (right).

Let us now illustrate how abduction can be used for spatial reasoning. Image understanding can be expressed as an explanatory process [5]. Observations can be images, or results of some image analysis process (e.g. segmentation of some structures in the images). The knowledge base models expert knowledge on the domain, on the structures present in the scene and on their relations (contrast, spatial relations...). It can be expressed in description logic for instance. A solution to this problem consists in translating knowledge and observation in a lattice of concepts and by applying erosions in this lattice to find the best explanation according to a minimality criterion [5]. Another algorithmic solution relies on tableau methods [99]. In the example in Fig.7, a MRI brain image with a tumor can be interpreted, at a higher level, using an anatomical knowledge base, as "Brain with a Peripheral Small Deforming Tumor".

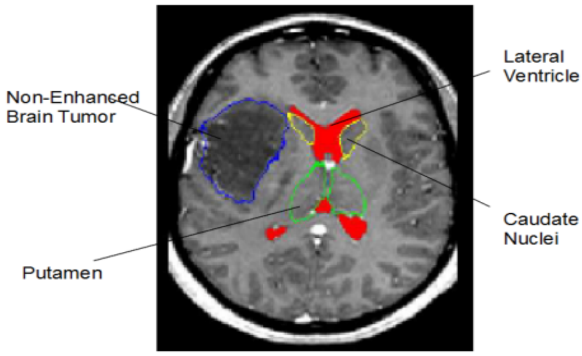

Figure 7. Pathological brain with a tumor.

Let us now consider modal morpho-logic, where the two modalities are defined as erosion and dilation (i.e. $\Box \equiv \varepsilon$ and $\Diamond \equiv \delta$), and consider the domain of mereotopology, specifically the Region Connection Calculus (RCC) formalism [81]. In this theory, several topological relations are defined from a connection predicate, in first order logic. Modal morpho-logic leads to simpler and decidable expressions of some of these relations. Let us provide a few examples, where φ and ψ are formulas representing abstract spatial entities:

- φ is a tangential part of ψ iff $\varphi \rightarrow \psi$ and $\Diamond\varphi \wedge \neg\psi \not\rightarrow \perp$ (or $\varphi \rightarrow \psi$ and $\varphi \wedge \neg\Box\psi \not\rightarrow \perp$). A simple model in the 2D space of such a relation is illustrated in Figure **??**.
- φ is a non tangential part of ψ iff $\Diamond\varphi \rightarrow \psi$ (or $\varphi \rightarrow \Box\psi$).
- φ and ψ are externally connected (adjacent) iff $\varphi \wedge \psi \rightarrow \perp$ and $\Diamond\varphi \wedge \psi \not\rightarrow \perp$ (or $\varphi \wedge \Diamond\psi \not\rightarrow \perp$).

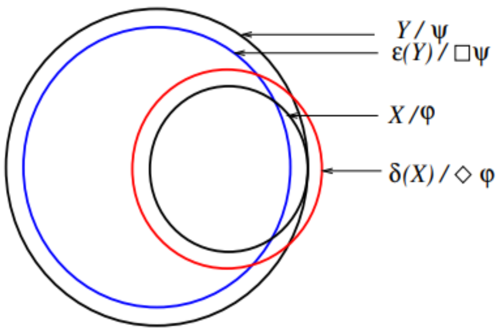

Figure 8. Tangential part from morphological operators (X and Y are models of formulas φ and ψ, respectively).

Further links between mathematical morphology and RCC can be found in [21, 25, 63], with extensions for fuzzy sets and bipolar fuzzy sets. These extensions are obtained by considering (bipolar) fuzzy objects and (bipolar) fuzzy connectives in the previous equations.

4.2. **Structural models.** A well-known problem in image and computer vision is the semantic gap between the physical level of images, that is features extracted by image processing, and symbols expressed in a language. On the one hand, linguistic descriptions of prior knowledge about the images and the domains can be translated into formal models and algorithms to guide image understanding. On the other hand, automatic annotation, that is generation of linguistic descriptions of image content, is an increasing, though recent, field of research that evolves rapidly. We present in this section a short overview of the state of the art in this domain[2], focusing on the use of fuzzy sets (a more detailed version can be found in [22]).

Image interpretation and understanding aim at global scene recognition, to obtain high-level descriptions of the objects in their context, including their spatial arrangement. Image understanding includes also semantic interpretation. Semantics is not present in the image itself, but requires some prior knowledge (for example expressed as formal models) to extract it. All this should then lead to verbal or linguistic descriptions of the image content. One main problem is the semantic gap, close to the symbol grounding problem. Fuzzy methods provide useful tools to deal with both ontological concepts (often provided as linguistic terms) and concrete domains, and to establish links between them, as explained in Section 3. Two main directions can be identified in connection with linguistic descriptions of images: (i) using linguistic descriptions expressed in a model to guide image interpretation (Section 4.2.1), (ii) deriving linguistic descriptions of images based on image features (Section 4.2.2). Our focus is on methods relying on fuzzy models (for representing vague knowledge, imprecision in images and in concepts, etc.), associated with symbolic and structural models.

4.2.1. *From linguistic descriptions to image understanding.* To go beyond individual object recognition, image understanding requires descriptions of the spatial organization of objects in images. In knowledge-based approaches, the models should then include such structural knowledge. Models have then to be combined with image information, to finally lead to scene understanding. As mentioned in Section 3, spatial relations can be modeled efficiently using fuzzy mathematical morphology, and then introduced as attributes or concepts in graphs, ontologies, conceptual graphs, which are formal computational models often used to guide image understanding.

Usually several spatial relations have to be combined to describe one or several objects, and then the variety of combination operators in fuzzy set theory offers a lot of flexibility in their choice [9], that can be adapted to any situation at hand, and which may deal with heterogeneous information [50, 98].

A survey of knowledge-based systems for image interpretation until 1997 can be found in [41]. Here we focus on more recent approaches, and using fuzzy formalisms, which can then be seen as halfway between purely quantitative and purely qualitative reasoning. A typical example in this domain concerns model-based structure recognition in images, where the model represents spatial entities and relations between them. Two main components of this domain are spatial knowledge representation and reasoning. In particular spatial relations constitute an important part of the knowledge we have to handle. Two types of questions arise when reasoning with spatial relations: (i) given two objects (possibly fuzzy), assess the degree to which a relation is satisfied; (ii) given one reference object, define the area of space in which a relation to this reference is satisfied (to some degree). It has been shown in [13] that the association of three frameworks in a unified way, namely mathematical morphology, fuzzy sets and logics, allows on the one hand matching two important requirements: expressiveness and completeness with respect to the types of spatial information to be represented [1], and on the other hand performing successful reasoning tasks for image understanding.

[2]Note that we do not deal with the generation of (synthetic) images based on linguistic descriptions in this paper.

A common computational representation of structural information to guide image interpretation consists of a graph, where vertices represent objects or image regions (possibly with attributes such as shape, size, color or gray level), and edges carry the structural information (spatial relations between objects, radiometric contrast between regions...). One type of approach consists in deriving a graph from the image itself, based on a preliminary segmentation of the image into homogeneous regions, and to express the recognition as a graph matching problem between the image graph and the model graph, which, however, raises combinatorial problems [35, 40]. In [67, 78] an initial labeling of the image regions is performed, and spatial relations are used to refine this labeling or to extract the objects of interest. Since achieving a correct initial segmentation is known to be a very difficult problem in image processing, no isomorphism usually exists between the graphs to be matched. This leads naturally to the need to find an inexact matching, for instance by allowing several image regions to be assigned to one model vertex, or by relaxing the notion of morphism to the one of fuzzy morphism [38, 79]. As an example, in [42], an over-segmentation of the image is used, which is easier to obtain. Fuzzy relations can then be used to get the final labeling and interpretation [52]. A model structure is then explicitly associated with a set of regions and the recognition is expressed as a constraint satisfaction problem. Some methods rely on fuzzy graph comparison and matching, using genetic algorithm, estimation of distributions algorithms, or graph kernels involving spatial relations. One of the main issues in these methods is the design of an appropriate objective function, guaranteeing that it is optimal for the right solution, which is a difficult task. Still relying on a preliminary segmentation, some approaches have been proposed, for instance using ontologies [60, 75], with fuzzy extensions, besides other types of methods (grammatical or probabilistic ones). Fuzzy Region Connection Calculus (RCC) [87] can also be used to identify objects based on their mereotopological relations, as done in the crisp case (e.g. [63, 64]).

To overcome the difficulty of obtaining a relevant segmentation, the segmentation and the recognition can also be performed simultaneously. For instance, the method proposed in [24, 39] consists in sequentially segmenting and recognizing each object of interest, in a pre-calculated order [53]. The objects that are easier to segment are considered first and taken as reference. Spatial relations to these reference objects encoded in the structural model, and formalized as fuzzy sets, are used as constraints to guide the segmentation and recognition of other objects. However the extraction of the first objects can be difficult if it is not sufficiently constrained, and due to the sequential nature of the process, the errors are potentially propagated. Backtracking may then be needed, as proposed in [53]. Similar approaches have been used for mobile robot navigation in [55], for vessel tracking in MRI in [92].

These approaches can be formalized also as ontological reasoning [60], where both an ontology of the domain can be enriched by fuzzy spatial relations. A first step consists in extracting information from the domain ontology by querying it. The next step consists in actually segmenting (and recognizing) the structure of interest.

To overcome the problems raised by sequential approaches, while avoiding the need of an initial segmentation, another method, still relying on a structural model, but solving the problem in a global way, was proposed in [74]. A solution is the assignment of a region of space to each model object, that satisfies the constraints expressed in the model. A solution is obtained by reducing progressively the solution domain for all objects by excluding assignments that are inconsistent with the structural model. Constraint networks [84] constitute an appropriate framework both for the formalization of the problem and for the optimization. This approach was extended in [94] to fuzzy constraint satisfaction problems (extending [47]) do deal with more complex relations, or involving an undetermined number of objects, and applied to the interpretation of high resolution remote sensing images.

Besides recognition and segmentation, fuzzy spatial relations and more generally fuzzy spatial information have proved useful for other interpretation tasks, such as multiple object tracking [96, 97], graph kernels for machine learning [2], facial expression understanding [80], navigation in unknown environments in robotics [36, 45, 56], among others.

4.2.2. *From image analysis to image content descriptions.* Let us now consider the other way around, where the objective is to start from image features to derive descriptions of the image content in a way as close as possible to natural language, using some terms (called "tags" in image annotation), or more recently whole sentences, for automatic image captioning. This refers to the typical "show and tell" approaches, that benefit from recent advances in machine learning (convolutional networks and deep learning), or use mostly clustering and probabilistic approaches [65, 85, 95]. As an example using fuzzy models, let us cite [7] where fuzzy multimedia ontologies were developed for semantic image annotation. Tags were identified based on consistency of candidate concepts, obtained from SVM classification, and tested using fuzzy description logic reasoning. Interesting methods using structural representations such as graphs or grammars are also worth to be mentioned [76, 93]. Although a large majority of approaches rely on probabilistic models or learning methods, some of them, in particular structural approaches using graphs or grammars, could be enhanced by fuzzy components to deal with imprecision, vagueness, variability. Still a few fuzzy approaches have been proposed, as described next.

One problem with neural networks is that it may be difficult to understand which rules or reasoning processes they have learned. This question was answered in [69] where satellite image classification was performed using fuzzy neural networks, producing also the fuzzy rules that are actually used by the system, and that are understandable by domain experts, thus providing a description of the image and of how this description was obtained. Fuzzy sets learned from neural networks were used in [66] in the domain of art image retrieval. The linguistic variables describe "fuzzy aesthetic semantics", in terms of action, relaxation, joy, fear, etc., associated with degrees of satisfaction. Using also a neural network, associated with a fuzzy classifier and an expert system reasoning on low level features, the work in [77] leads to descriptions of facial expressions. Fuzzy rules were also exploited to generate simple linguistic descriptions of image content [3]. This approach was used for various applications, such as circular structures on Mars, traffic, human gait, medical images... In [8], a clustering and compression method was proposed to provide a small number of fuzzy rules having a linguistic meaning, which constitute fuzzy models that provide linguistic descriptions of low-level features in images. Applications in matching were developed.

At a more structural level, image descriptions involve spatial relations. For instance, in [37], linguistic features describing regions were obtained by fuzzy segmentation, fuzzy spatial relations and locations. From a set of predefined linguistic terms, a brief and accurate description of the whole image is then generated. In [71, 90], previous work by the authors on computation of relative direction was used to derive linguistic descriptions of relative positions in images, associated with a qualitative validity of the description. A typical example of result is "the building is perfectly to the right of the reference object; the description is satisfactory".

Another source of inspiration may come from work on summarization. For instance, in [86] the summarization of image databases was based on low-level features and fuzzy labels. While summarization was not much addressed until now for images, several works have emerged for time series and signals, see for instance the special issues on linguistic description of time series in [68], with several papers on automatic generation of linguistic descriptions of data, mapping from non linguistic to linguistic expressions, linguistic summarization. Although some ideas and methods could probably be exploited, the problem when dealing with images is quite different, and there is some work to do to really account for the spatial nature of images and for structural information and knowledge.

One issue in all these approaches is the validation of the obtained linguistic descriptions of the images. Most of the time, a simple comparison with the description provided by a human is performed. This assumes defining a common vocabulary and language, which raises the issue of the level of the description. For instance, describing a brain image may take different forms depending on whether it is intended for a wide public audience, for a patient or for an neurology expert, ranging thus from "an abnormal structure is present in the brain", to "a peripheral non-enhanced tumor is present in the right hemisphere" (see the example in Section 4.1, using abductive reasoning). The example in [99] exploits the two directions described in this section. Starting from a linguistic description of the expert knowledge (here neuro-anatomy), a formal model is built (an ontology) and a knowledge base is derived in description logics, which will guide the image interpretation. At the end of the interpretation process, the result is expressed in the same logics, close to the expert natural language.

Fuzzy sets are then involved both at the representation level, to cope with imprecision in knowledge and data, and at the reasoning level. For instance, deriving that a tumor is large, or close to a given anatomical structure, calls for comparison of fuzzy sets representing the concepts (close, large) on the one hand, and the information derived from the image (e.g. as fuzzy numbers or distributions) on the other hand. To this end, methods based on mathematical morphology and optimal transport have been proposed in [23].

5. Conclusion

In this paper, we have shown that the mathematical bases of mathematical morphology, applied to fuzzy sets, lead to an algebraic framework guaranteeing good formal properties, very intuitive interpretations, and suitable for further modeling of spatial relations and reasoning. Hints on the use of this framework for spatial reasoning and image understanding have been summarized as well.

Two current important trends in artificial intelligence are hybrid AI and interpretability or explainability of AI methods. Interestingly enough, fuzzy approaches contribute to these two trends. First, as shown in this paper, fuzzy sets can be combined with other aspects of AI, such as logics, structural models, spatial reasoning, hence referring to hybrid AI. Secondly, the link between symbolic knowledge and numerical information provided by linguistic variables and concrete domains strongly facilitates interpretability. Finally, abductive reasoning directly aims at finding explanations.

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Isabelle Bloch - graduated from the Ecole des Mines de Paris, Paris, France, in 1986, and received the master's degree from the University Paris 12, Paris, in 1987, the Ph.D. degree from the Ecole Nationale Suprieure des Tlcommunications (Tlcom Paris), Paris, in 1990, and the Habilitation degree in mathematics and computer science from University Paris 5, Paris, in 1995. She has been a Professor at Tlcom Paris until 2020 and is now a Professor at Sorbonne Universit.

Her current research interests include 3D image understanding, mathematical morphology, information fusion, fuzzy set theory, structural, graph-based, and knowledge-based object recognition, spatial reasoning, symbolic and hybrid artificial intelligence, and medical imaging.